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Integral equation approach to reflection and transmission of a plane TE-wave at a (linear/nonlinear) dielectric film with spatially varying permittivity

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Abstract

A method is proposed for obtaining certain solutions (TE-polarized electromagnetic waves) of the Helmholtz equation, describing the reflection and transmission of a plane monochromatic wave at a (linear or nonlinear) dielectric film situated between two linear semi-infinite media. All three media are assumed to be lossless, nonmagnetic and isotropic. The permittivity of the film is modelled by (i) a continuously differentiable real-valued function of the transverse coordinate, and by (ii) a Kerr-nonlinearity. It is shown that the solution of the Helmholtz equation exists in the form of a uniformly convergent series (in case (i)) and in the form of a uniformly convergent sequence (in case (ii)) of iterations of the equivalent Volterra integral equation. Numerical results of the approach are presented.

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1. Introduction

Many problems of optics involve the study of the optical response of a dielectric film with a specific permittivity. For constant permittivity the problem is of particular interest in linear optics [1]. For arbitrary varying field intensity independent permittivity, there exists (to our knowledge) no general solution to Maxwell's equations. In this case, traditionally the transfer matrix approach [2] is used discretizing the film by a number of plane parallel dielectric slabs of infinitesimal thickness with constant permittivity. The optical response of each slab is described by a 2×2 matrix and the net response of the film is obtained through matrix

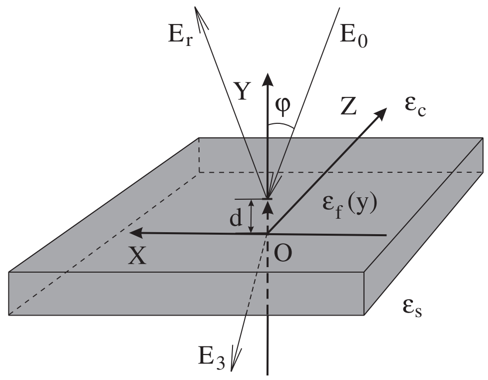


Figure 1. Configuration considered in this paper. A plane wave is incident to a linear slab (situated between two linear media) to be reflected and transmitted.

multiplication. Furthermore the Green function method [3], the invariant embedding approach [4] and the wave splitting theory [5] are well-known techniques in this respect. Recently, an iterative approach, based on a pair of coupled differential equations generated from Maxwell's equations was proposed [6].

In nonlinear optics, the Kerr-like nonlinear dielectric film has been the focus of a number of studies [7–12]. With respect to the nonlinear Fabry–Perot system, the present problem has been approached under special conditions by several authors: Marburger and Felber [13] simplified the analysis by imposing boundary conditions which suppose the nonlinear slab is separated from the linear media by perfect mirrors. Danikaert *et al* [14] treated the steady-state response of a nonlinear Fabry–Perot resonator including nonlinear absorption and oblique incidence for transverse-electric and transverse-magnetic polarized fields. Haeltermann *et al* [15] and Vitrant [16] presented a unified nonlinear theory for transverse effects of Fabry–Perot resonators simplifying numerical calculations and providing a good understanding of optical bistability.

In this paper we suggest an alternative approach to the problem of scattering from a dielectric film with a permittivity $\varepsilon_f(y) = \widehat{\varepsilon}_f(y) + a|E|^2$, where $\widehat{\varepsilon}_f(y)$ is a continuously differentiable real-valued function of the transverse coordinate y and $a|E|^2$ denotes the usual local Kerr-nonlinearity with real a. We transform the Helmholtz equation valid for the film to a Volterra integral equation and solve the latter by iteration subject to the appropriate boundary conditions.

The paper is organized as follows. In sections 2 and 3 the linear case ($a = 0$) is considered. In section 2 we reduce the Helmholtz equation to a Volterra integral equation for the electric field intensity and to a quadrature determining the phase of the electric field. The method is applied to the transmission case in section 3. In section 4 the nonlinear case is considered and compared with the exact solution if $\widehat{\varepsilon}_f(y) = \text{const}$. The last section contains the summary and a short outlook.

2. Reduction of the problem to a Volterra integral equation

Referring to figure 1 we consider the reflection and transmission of an electromagnetic plane wave at a dielectric film between two linear semi-infinite media (substrate and cladding). All media are assumed to be homogeneous in x- and z-direction, nonabsorbing, isotropic and non-magnetic. The permittivity of the film is assumed to be characterized by a function $\varepsilon_f(y)$.

A plane wave of frequency ω_0 and intensity E_0^2, with electric vector $\mathbf{E_0}$ parallel to the z-axis (TE) is incident on the film of thickness d. Since the geometry is independent of the z-coordinate and because of the supposed TE-polarization fields are parallel to the z-axis ($\mathbf{E} = \mathbf{E}_z$). We look for solutions of Maxwell's equations that satisfy the boundary conditions (continuity of $\mathbf{E}_z$ and $\partial\mathbf{E}_z/\partial y$ at interfaces $y \equiv 0$ and $y \equiv d$). Due to the requirement of the transactional invariance in x-direction and partly satisfying the boundary conditions the fields tentatively are written as ($\hat{z}$ denotes the unit vector in z-direction)

$$\mathbf{E}(x, y, t) = \begin{cases} \hat{z}\frac{1}{2}[E_0\,\mathrm{e}^{\mathrm{i}(px - q_c(y-d) - \omega_0 t)} + E_r\,\mathrm{e}^{\mathrm{i}(px + q_c\cdot(y-d) - \omega_0 t)} + \text{c.c.}] & y > d \\ \hat{z}\frac{1}{2}[E(y)\,\mathrm{e}^{\mathrm{i}(px + \vartheta(y) - \omega_0 t)} + \text{c.c.}] & 0 < y < d \\ \hat{z}\frac{1}{2}[E_3\,\mathrm{e}^{\mathrm{i}(px - q_s y - \omega_0 t)} + \text{c.c.}] & y < 0 \end{cases} \tag{1}$$

where $E(y)$, $p = \sqrt{\varepsilon_c}k_0 \sin\varphi, q_c$, and $\vartheta(y)$ are real and $E_r = |E_r|\exp(\mathrm{i}\delta_r)$ and $E_3 = |E_3|\exp(\mathrm{i}\delta_t)$ are independent of y. The parameter q_s is assumed to be real (transmission case) in the following. We exclude purely imaginary q_s (total reflection case)[3].

The permittivity is modelled by

$$\varepsilon(y) = \begin{cases} \varepsilon_c & y > d \\ \varepsilon_f(y) = \widehat{\varepsilon}_f(y) & 0 < y < d \\ \varepsilon_s & y < 0 \end{cases} \tag{2}$$

with real constants $\varepsilon_c, \varepsilon_s$ and with $\widehat{\varepsilon}_f$ as a real continuously differentiable function of y on $[0, d]$. By inserting (1) and (2) into Maxwell's equations we obtain the linear Helmholtz equations, valid in each of the three media ($j = s, f, c$),

$$\frac{\partial^2 \tilde{E}_j(x, y)}{\partial x^2} + \frac{\partial^2 \tilde{E}_j(x, y)}{\partial y^2} + k_0^2\varepsilon_j\tilde{E}_j(x, y) = 0 \qquad j = s, f, c \tag{3}$$

where $k_0^2 = \omega_0^2/c^2$ and $\tilde{E}_j(x, y)$ denotes the time-independent part of $\mathbf{E}(x, y, t)$.

Scaling x, y, z, p, q_c, q_s by the wavelength λ_0 and ε by ε_0, respectively, equation (3) reads

$$\frac{\partial^2 \tilde{E}_j(x, y)}{\partial x^2} + \frac{\partial^2 \tilde{E}_j(x, y)}{\partial y^2} + 4\pi^2\varepsilon_j\tilde{E}_j(x, y) = 0 \qquad j = s, f, c \tag{4}$$

where the same symbols have been used for unscaled and scaled quantities. Using ansatz (1) in equation (4) we get for the semi-infinite media

$$q_j^2 = 4\pi^2\varepsilon_j - p^2 \qquad j = s, c. \tag{5}$$

For the film ($j = f$), we obtain

$$\frac{\mathrm{d}^2 E(y)}{\mathrm{d}y^2} - E(y)\left(\frac{\mathrm{d}\vartheta(y)}{\mathrm{d}y}\right)^2 + [4\pi^2\widehat{\varepsilon}_f(y) - p^2]E(y) = 0 \tag{6}$$

and

$$E(y)\frac{\mathrm{d}^2\vartheta(y)}{\mathrm{d}y^2} + 2\frac{\mathrm{d}\vartheta(y)}{\mathrm{d}y}\frac{\mathrm{d}E(y)}{\mathrm{d}y} = 0. \tag{7}$$

Equation (7) can be integrated leading to

$$E^2(y)\frac{\mathrm{d}\vartheta(y)}{\mathrm{d}y} = c_1 \tag{8}$$

[3] In this case the constants c_1, c_2 change because equations (22) and (23) must be changed according to the boundary conditions.

where c_1 is a constant that has to be determined by means of the boundary conditions. Insertion of $\mathrm{d}\vartheta/\mathrm{d}y$ into equation (6) yields

$$\frac{\mathrm{d}^2 E(y)}{\mathrm{d}y^2} + q_f^2(y)E(y) - \frac{c_1^2}{E^3(y)} = 0 \tag{9}$$

with

$$q_f^2(y) = 4\pi^2\widehat{\varepsilon}_f(y) - p^2. \tag{10}$$

As will be shown below, real q_s (transmission) implies $c_1 \neq 0$. Introducing $I(y) = E^2(y)$, multiplying equation (9) by $4E^3(y)$, and differentiating the result with respect to y we obtain

$$\frac{\mathrm{d}^3 I(y)}{\mathrm{d}y^3} + 4\frac{\mathrm{d}\big(q_f^2(y)I(y)\big)}{\mathrm{d}y} = 2\frac{\mathrm{d}\big(q_f^2(y)\big)}{\mathrm{d}y}I(y). \tag{11}$$

Representing $\widehat{\varepsilon}_f(y)$ in the form $\widehat{\varepsilon}_f(y) = \varepsilon_f^0 + \widetilde{\varepsilon}_f(y)$, where ε_f^0 is a constant, equation (11) can be integrated to yield

$$\frac{\mathrm{d}^2 I(y)}{\mathrm{d}y^2} + 4\kappa^2 I(y) = -16\pi^2\widetilde{\varepsilon}_f(y)I(y) + 8\pi^2\int_0^y \frac{\mathrm{d}\widetilde{\varepsilon}_f(\tau)}{\mathrm{d}\tau}I(\tau)\mathrm{d}\tau + c_2 \tag{12}$$

where $\kappa^2 = 4\pi^2\varepsilon_f^0 - p^2$ and c_2 denotes another constant of integration. The homogeneous equation $\mathrm{d}^2 I(y)/\mathrm{d}y^2 + 4\kappa^2 I(y) = 0$ has the general solution

$$\tilde{I}_0(y) = A\cos(2\kappa y) + B\sin(2\kappa y) \tag{13}$$

so that the solution of equation (12) reads [17]

$$I(y) = \tilde{I}_0(y) + \int_0^y \mathrm{d}t\,\frac{\sin 2\kappa(y-t)}{2\kappa}\left(c_2 + 8\pi^2\int_0^t \mathrm{d}\tau\,\frac{\mathrm{d}\widetilde{\varepsilon}_f(\tau)}{\mathrm{d}\tau}I(\tau) - 16\pi^2\widetilde{\varepsilon}_f(t)I(t)\right) \tag{14}$$

where the constant c_2 must be determined by the boundary conditions. The Volterra equation (14) is equivalent to equation (3) for $0 < y < d$. According to equation (14) $I(y)$ and $\tilde{I}_0(y)$ satisfy the boundary conditions at $y = 0$. Evaluating the first integral on the right-hand side, equation (14) can be written as

$$I(y) = \tilde{I}_0(y) + \frac{c_2}{2\kappa^2}\sin^2(\kappa y) + \int_0^y K(y,t)I(t)\,\mathrm{d}t \tag{15}$$

with

$$K(y,t) = -8\pi^2\frac{\sin 2\kappa(y-t)}{\kappa}\widetilde{\varepsilon}_f(t) - 2\pi^2\frac{\cos 2\kappa(y-t) - 1}{\kappa^2}\frac{\mathrm{d}\widetilde{\varepsilon}_f(t)}{\mathrm{d}t}. \tag{16}$$

The solution of equation (15) can be represented as a uniformly convergent series of iterations (cf appendix A):

$$I(y) = \sum_{j=0}^{\infty} I_j(y) \tag{17}$$

$$I_j(y) = \int_0^y K(y,t)I_{j-1}(t)\mathrm{d}t \qquad j = 1, 2, \ldots \tag{18}$$

where

$$I_0(y) = \tilde{I}_0(y) + \frac{c_2}{2\kappa^2}\sin^2(\kappa y). \tag{19}$$

With the solution $I(y)$, determined by equation (17), the phase function $\vartheta(y)$ is given, according to equation (8), by

$$\vartheta(y) = \vartheta(d) + c_1\int_d^y \frac{\mathrm{d}\tau}{I(\tau)}. \tag{20}$$

3. Transmission and reflection at a linear dielectric film

3.1. Boundary conditions and associated relations

We consider the case of real q_s. Continuity of $\mathbf{E}(y)$ and $\mathrm{d}\mathbf{E}(y)/\mathrm{d}y$ at $y = 0$ and $y = d$ implies

$$E(0) = E_3\,\mathrm{e}^{-\mathrm{i}\vartheta(0)} \tag{21}$$

$$\left.\frac{\mathrm{d}E(y)}{\mathrm{d}y}\right|_{y=0} = 0 \tag{22}$$

$$\left.\frac{\mathrm{d}\vartheta(y)}{\mathrm{d}y}\right|_{y=0} = -q_s \tag{23}$$

$$E_0 + E_r = E(d)\,\mathrm{e}^{\mathrm{i}\vartheta(d)} \tag{24}$$

$$2E_0\,\mathrm{e}^{-\mathrm{i}\vartheta(d)} = \frac{\mathrm{i}}{q_c}\left.\frac{\mathrm{d}E(y)}{\mathrm{d}y}\right|_{y=d} + E(d)\left(1 - \frac{1}{q_c}\left.\frac{\mathrm{d}\vartheta(y)}{\mathrm{d}y}\right|_{y=d}\right). \tag{25}$$

According to equations (8), (21) and (23) the constant c_1 is given by

$$c_1 = -q_s|E_3|^2 = -q_s I(0). \tag{26}$$

Using equations (9), (12), (21) and (22), the constant of integration c_2 in equation (14) is determined by

$$c_2 = 2|E_3|^2\big(q_s^2 + q_f^2(0)\big) = 2I(0)\big(q_s^2 + q_f^2(0)\big). \tag{27}$$

Equations (9), (24), (25), (27) imply

$$2E_0(E_0 + \mathrm{Re}(E_r)) = E^2(d) + \frac{q_s}{q_c}|E_3|^2 \tag{28}$$

$$E^2(d) = (E_0 + \mathrm{Re}(E_r))^2 + (\mathrm{Im}(E_r))^2 \tag{29}$$

and hence

$$|E_r|^2 = E_0^2 - \frac{q_s}{q_c}|E_3|^2 \tag{30}$$

or, in terms of reflectivity R and transmissivity T,

$$R = 1 - \frac{q_s|E_3|^2}{q_c E_0^2} = 1 - T \tag{31}$$

where $R = \frac{|E_r|^2}{E_0^2}$, $T = \frac{q_s|E_3|^2}{q_c E_0^2}$.

By considering the imaginary parts of equations (24) and (25) one obtains

$$\mathrm{Im}(E_r) = -\frac{E(d)\left.\frac{\mathrm{d}E(y)}{\mathrm{d}y}\right|_{y=d}}{2q_c E_0} \tag{32}$$

which leads to, taking into account equations (29), (28), (32),

$$E_0^2 = \frac{\left(\left.\frac{\mathrm{d}I(y)}{\mathrm{d}y}\right|_{y=d}\right)^2 + 4(q_c I(d) + q_s|E_3|^2)^2}{16 I(d) q_c^2}. \tag{33}$$

The phase $\vartheta(d)$ can be determined by evaluation of equation (25) to give

$$\sin\vartheta(d) = -\frac{\left.\frac{\mathrm{d}I(y)}{\mathrm{d}y}\right|_{y=d}}{4q_c E_0\sqrt{I(d)}}. \tag{34}$$

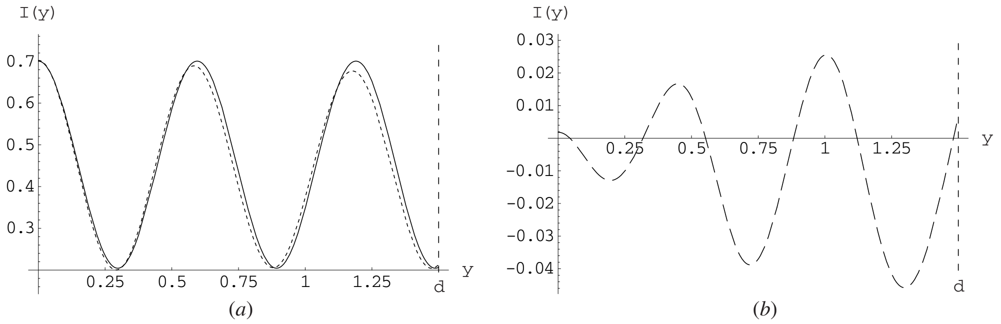


Figure 2. (*a*) Dependence of the field intensity $I(y)$ (first iteration) inside the slab on the transverse coordinate y for $\varepsilon_c = \varepsilon_s = 1, \varepsilon_f^0 = 1.5, \varphi = 0.35\pi, E_0^2 = 1, d = 1.5$. The dashed curve corresponds to the periodic dependence of $\varepsilon_f(y) = \varepsilon_f^0 + \delta\cos^2 b(\frac{y}{d})$ for $\delta = \frac{1}{30}, b = 10$. The full curve corresponds to the case of constant permittivity $\varepsilon_f(y) = \varepsilon_f^0$ ($\delta = 0$); (*b*) the difference between the intensities from (*a*).

The phase shift on transmission δ_t is equal to $\vartheta(0)$, the phase shift on reflection is determined by equations (24) and (30) as

$$\sin\delta_r = -\frac{\frac{\mathrm{d}I(y)}{\mathrm{d}y}\Big|_{y=d}}{4q_c E_0^2\sqrt{1-\frac{q_s I(0)}{q_c E_0^2}}}. \tag{35}$$

3.2. Solutions

Introducing $\widehat{I}(y) = I(y)/I(0)$ and using the relations of the foregoing subsection the normalized intensity $\widehat{I}(y)$ and the phase $\vartheta(y)$ can be written as

$$\widehat{I}(y) = \cos(2\kappa y) + \frac{q_s^2 + q_f^2(0)}{\kappa^2}\sin^2(\kappa y) + \int_0^y K(y,t)\widehat{I}(t)\,\mathrm{d}t \tag{36}$$

where equations (13), (21), (22), (27) have been used, and, taking equations (20), (26) and (34) into account,

$$\vartheta(y) = -\arcsin\frac{\frac{\mathrm{d}\widehat{I}(y)}{\mathrm{d}y}\Big|_{y=d}}{\sqrt{\left(\frac{\mathrm{d}\widehat{I}(y)}{\mathrm{d}y}\Big|_{y=d}\right)^2 + 4(q_c\widehat{I}(d)+q_s)^2}} + q_s\int_y^d \frac{\mathrm{d}\tau}{\widehat{I}(\tau)}\,\mathrm{d}\tau. \tag{37}$$

Equations (31), (33) together with equations (36), (37) allow the optical response of the linear film to be calculated for arbitrary thickness d, arbitrary angles of incidence φ and arbitrary permittivity $\widehat{\varepsilon}_f(y)$. Equations (33) and (36) constitute a generalization of Fresnel's formulae in linear optics [18].

3.3. A numerical example

To illustrate the foregoing analysis we assume a periodic dependence of $\varepsilon_f(y)$ such that $\widetilde{\varepsilon}_f(y) = \delta\cos^2 b(y/d)$.

The first and the second iterations of (36) lead to expressions for $I(y)$ and $\vartheta(y)$, the corresponding field intensity inside the slab is shown in figures 2–4. In figure 5 the phase $\vartheta(y)$ is plotted.

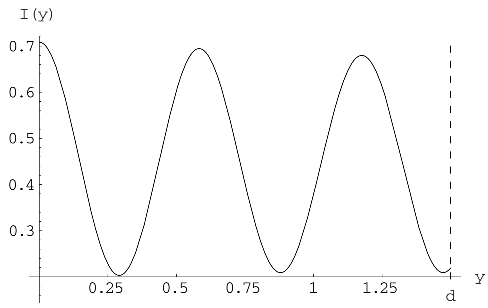


Figure 3. Dependence of the field intensity $I(y)$ (second iteration) inside the slab on the transverse coordinate y for the same parameters as in figure 2.

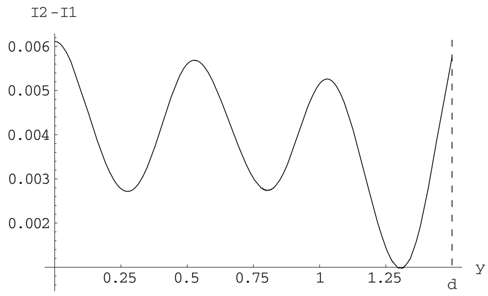


Figure 4. Difference between the field intensities after the first and second iterations.

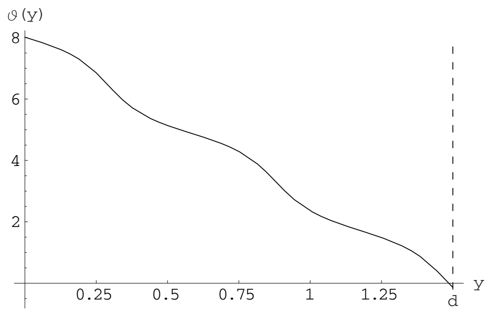
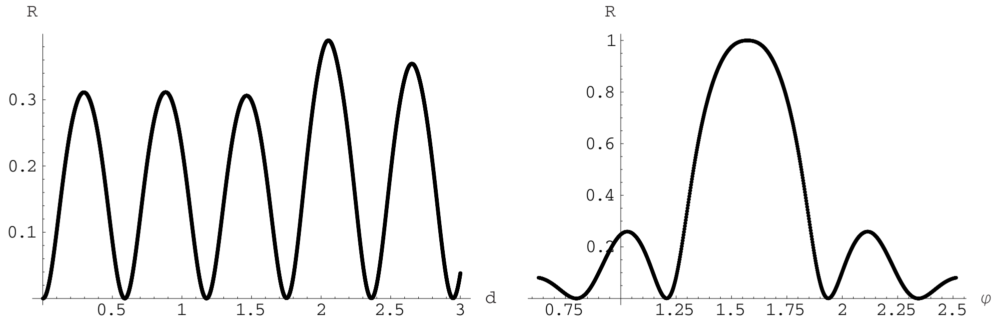


Figure 5. Phase $\vartheta(y)$ inside the film, parameters as in figure 2.

Using equation (33) the reflectivity R is given by

$$R = 1 - \frac{16 q_c q_s \widehat{I}(d)}{\left(\frac{\mathrm{d}\widehat{I}(y)}{\mathrm{d}y}\Big|_{y=d}\right)^2 + 4(q_c \widehat{I}(d) + q_s)^2}. \tag{38}$$

Plots of R are presented in figure 6. The character of the obtained dependence of R on the problem's parameters (thickness d, angle of incidence φ) agrees in general with the ones obtained for periodic layers [19]. The region, where $R \approx 1$ is analogous to Bragg reflection, well known in the dynamical theory of x-ray reflection [20].



(*a*) (*b*)

Figure 6. (*a*) Dependence of the reflectivity R on the layer thickness d for the same parameters as in figure 2; (*b*) dependence of the reflectivity R on the angle of incidence φ for the same parameters as in figure 2.

4. Transmission and reflection at a Kerr-like nonlinear dielectric film

We consider again the transmission case (q_s real) and assume a nonlinearity of the permittivity according to

$$\varepsilon_f = \varepsilon_f^0 + \widetilde{\varepsilon}_f(y) + aE^2(y) \qquad 0 < y < d \tag{39}$$

with real constant a. Using the same arguments as in section 2 we get in place of equation (6)

$$\frac{\mathrm{d}^2 E(y)}{\mathrm{d}y^2} - E(y)\left(\frac{\mathrm{d}\vartheta(y)}{\mathrm{d}y}\right)^2 + \left[4\pi^2\left(\left(\varepsilon_f^0 + \widetilde{\varepsilon}_f(y)\right) + aE^2(y)\right) - p^2\right] E(y) = 0. \tag{40}$$

In place of equations (15) and (16) we now obtain

$$I(y) = \tilde{I}_0(y) + \frac{c_2}{2\kappa^2}\sin^2(\kappa y) + \int_0^y K(y, t, I(t)) I(t)\,\mathrm{d}t \tag{41}$$

with

$$K(y, t, I(t)) = -\frac{\sin 2\kappa(y-t)}{\kappa}(8\pi^2\widetilde{\varepsilon}_f(t) + 6\pi^2 aI(t)) - 2\pi^2\frac{\cos 2\kappa(y-t) - 1}{\kappa^2}\frac{\mathrm{d}\widetilde{\varepsilon}_f(t)}{\mathrm{d}t}. \tag{42}$$

The solution of the nonlinear integral equation (41) can be represented as a limit of the uniformly convergent sequence $I_j(y)$ (cf appendix B)

$$I(y) = \lim_{j\to\infty} I_j(y) \tag{43}$$

$$I_j(y) = I_0(y) + \int_0^y K(y, t, I_{j-1}(t)) I_{j-1}(t)\,\mathrm{d}t \qquad j = 1, 2, \ldots \tag{44}$$

where $I_0(y)$ is given by equation (19). The uniform convergence is proved using the Banach fixed-point theorem. The condition for its applicability leads to a condition for the parameters of the problem (definitions of $\|N_1\|$, $\|N_2\|$, $\|I_0\|$, see appendices A and B)

$$\|N_1\| + 2\sqrt{\|N_2\| \cdot \|I_0\|} < 1. \tag{45}$$

Instead of equation (27) we obtain

$$c_2 = 2I(0)\left(q_s^2 + q_f^2(0) + 2\pi^2 aI(0)\right) \tag{46}$$

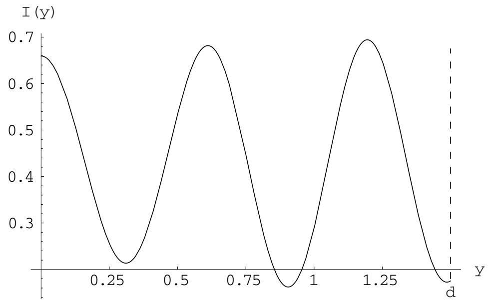


Figure 7. Dependence of the field intensity $I(y)$ (first iteration) inside the slab on the transverse coordinate y for $a = 0.01$, other parameters are the same as in figure 2.

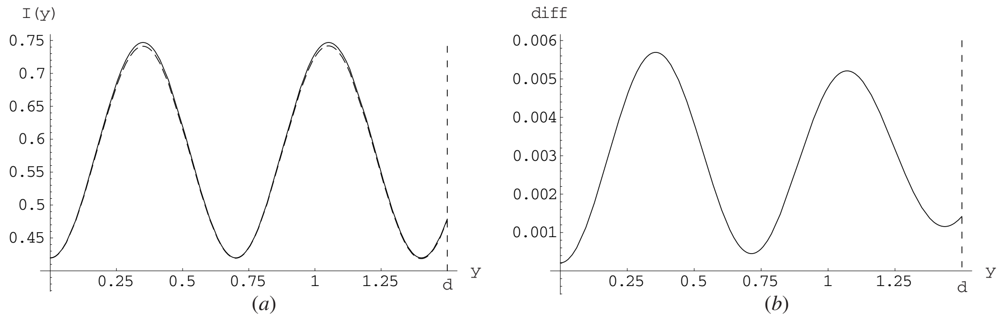


Figure 8. (*a*) Dependence of the field intensity $I(y)$ inside the slab on the transverse coordinate y for $\varepsilon_c = 1, \varepsilon_s = 1.7, \varepsilon_f^0 = 1.3, \varphi = 63.5^\circ, b = 10, \delta = 0, E_0^2 = 1, d = 1.5, a = 0.01$. Solid curve corresponds to the exact solution and dashed to the first iteration of equation (47); (*b*) the difference 'diff' between the curves from (*a*).

so that equation (41) reads

$$I(y) = I(0)\cos(2\kappa y) + \frac{\left(q_s^2 + q_f^2(0) + 2\pi^2 a I(0)\right)I(0)}{\kappa^2}\sin^2(\kappa y) + \int_0^y K(y, t, I(t))I(t)\,\mathrm{d}t \tag{47}$$

with $I(0)(= |E_3|^2)$ related to E_0^2 according to equation (33). The phase $\vartheta(y)$ is given by

$$\vartheta(y) = -\arcsin\frac{\left(\frac{1}{\sqrt{I(y)}}\frac{\mathrm{d}I(y)}{\mathrm{d}y}\right)\Big|_{y=d}}{4q_c E_0} + \int_y^d \frac{q_s I(0)}{I(\tau)}\,\mathrm{d}\tau. \tag{48}$$

To illustrate the procedure we assume the same periodic dependence of $\widehat{\varepsilon}_f(y)$ and the same parameters as for the linear case. The first iteration of equation (47) is shown in figure 7. For the special case $\delta = 0$ the results of the present method can be compared with the exact solution [21] (cf figure 8).

By means of a parametric plot the reflectivity R and the phase on reflection δ_r can be evaluated straightforwardly. Results are depicted in figures 9 and 10.

5. Summary and outlook

We have presented an iterative approach to solve the Helmholtz equation for a dielectric film with a permittivity according to $\varepsilon_f(y) = \widehat{\varepsilon}_f(y) + a|E|^2$. The solutions for the linear

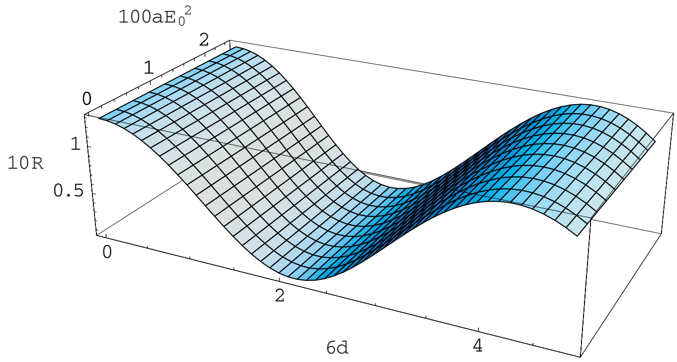


Figure 9. Dependence of the reflectivity R on aE_0^2 and on the thickness d for $\varepsilon_c = 1$, $\varepsilon_s = 1.7$, $\varepsilon_f^0 = 1.3$, $\varphi = 63.5°$, $\delta = 0$.

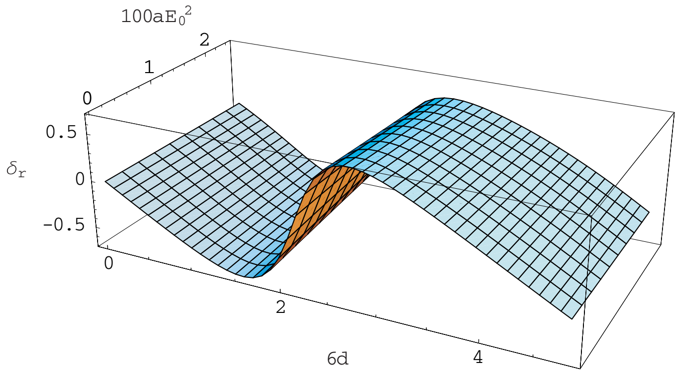


Figure 10. Dependence of the phase of reflection δ_r on aE_0^2 and d for the same parameters as in figure 9.

case ($a = 0$) and the nonlinear case ($a \neq 0$) have been expressed in terms of a uniformly convergent series and a uniformly convergent sequence of iterations of the Volterra equation, respectively. The main emphasis of the paper was on the derivation of the relationship between the Helmholtz equation and the Volterra integral equation and the proofs of convergence. Using the lowest-order iterations, analytical solutions and numerical results have been obtained straightforwardly. In particular, for the nonlinear case, the quality of the iteration scheme was estimated by comparison with the exact solution if $\widehat{\varepsilon}_f(y)$ is a constant (section 4) [21].

In any physically realizable film, absorption is present. Thus, it would be most intriguing to apply the above method to an absorbing film by assuming $\widehat{\varepsilon}_f(y)$ to be complex. But this represents a nontrivial extension of the procedure. Together with the investigation of a purely imaginary q_s (total reflection) this will be a topic of a future paper.

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Appendix A. Proof of the uniform convergence of series (17) by mathematical induction

Denoting the norm by $\|I_0\| = \max_{0\leqslant y\leqslant d} |I_0(y)|$ and $\|K\| = \max_{0\leqslant y,t\leqslant d} |K(y,t)|$ the iterations $I_j(y)$ can be estimated according to

$$|I_j(y)| \leqslant \|I_0\| \|K\|^j \frac{y^j}{j!}. \tag{A.1}$$

For $j = 1$ equation (18) implies

$$|I_1(y)| \leqslant \|I_0\| \int_0^y |K(y,t)| \mathrm{d}t \leqslant \|I_0\| \|K\| y. \tag{A.2}$$

Assuming that (A.1) holds one obtains

$$|I_{j+1}(y)| \leqslant \|I_0\| \|K\|^{j+1} \int_0^y \frac{t^j}{j!}\, \mathrm{d}t. \tag{A.3}$$

Thus (A.1) is valid for all j, leading to

$$|I(y)| \leqslant \sum_{j=0}^{\infty} |I_j(y)| \leqslant \|I_0\| \sum_{j=0}^{\infty} \frac{(\|K\|y)^n}{n!} = \|I_0\|\, \mathrm{e}^{y\|K\|}. \tag{A.4}$$

Series (17) converges uniformly on $[o, d]$.

Appendix B. Proof of the uniform convergence of sequence (43)

We consider the nonlinear operator F

$$F(I) := I_0(y) + N_1 I + N_2 I^2 \tag{B.1}$$

where N_1 and N_2 are linear bounded integral operators in the Banach space $C[o, d]$ and $I_0(y)$ is given by equation (19).

$$N_1\psi := \int_0^y K_1\psi(t)\, \mathrm{d}t \qquad N_2\varphi := \int_0^y K_2\varphi(t)\, \mathrm{d}t \tag{B.2}$$

with

$$K_1 = \left(-\frac{\sin 2\kappa(y-t)}{\kappa} 8\pi^2 \widetilde{\varepsilon}_f(t) - 2\pi^2 \frac{\cos 2\kappa(y-t) - 1}{\kappa^2} \frac{\mathrm{d}\widetilde{\varepsilon}_f(t)}{\mathrm{d}t}\right)$$

$$K_2 = \left(-\frac{6\pi^2 a}{\kappa} \sin 2\kappa(y-t)\right). \tag{B.3}$$

The norms $\|N_1\|$, $\|N_2\|$ are defined by

$$\|N_1\| = \max_{0\leqslant y,t\leqslant d} \int_0^y |K_1|\, \mathrm{d}t \qquad \|N_2\| = \max_{0\leqslant y,t\leqslant d} \int_0^y |K_2|\, \mathrm{d}t. \tag{B.4}$$

Then equation (41) can be rewritten in the operator form

$$I(y) = F(I)(y). \tag{B.5}$$

In order to prove that equation (B.5) under certain assumptions has only one solution, we consider the following quadratic equation:

$$z = \|I_0\| + \|N_1\| z + \|N_2\| z^2 \tag{B.6}$$

where $\|I_0\|$ is defined in appendix A. This equation has two positive roots if and only if the following conditions are satisfied:

$$(\|N_1\| - 1)^2 - 4\|N_2\| \|I_0\| > 0 \qquad \|N_1\| < 1. \tag{B.7}$$

These inequalities imply

$$\|N_1\| + 2\sqrt{\|N_2\| \cdot \|I_0\|} < 1. \tag{B.8}$$

Let r and R be the smallest and the largest roots of equation (B.6), respectively. In order to satisfy the conditions of the Banach fixed-point theorem [22] we have to check whether operator F maps the ball $S_R(0) = \{y \in C[0, d] : \|y\| < R\}$ (and $S_r(0)$) to itself. If $I(y) \in S_R(0)$ then

$$\|F(I)\| \leqslant \|I_0\| + \|N_1\|\|I\| + \|N_2\|\|I\|^2 < \|I_0\| + \|N_1\|R + \|N_2\|R^2 = R. \tag{B.9}$$

Thus $F(I) \in S_R(0)$. Hence equation (B.5) has at least one solution inside $S_R(0)$. F is contractive [22] in $S_r(0)$, because, if $I_1, I_2 \in S_r(0)$, then

$$\begin{aligned}\|F(I_1) - F(I_2)\| &= \|N_1(I_1 - I_2) + N_2(I_1^2 - I_2^2)\| \\ &\leqslant \|N_1\|\|I_1 - I_2\| + \|N_2\|\|I_1 - I_2\|\|I_1 + I_2\| \\ &\leqslant \|N_1\|\|I_1 - I_2\| + 2r\|N_2\|\|I_1 - I_2\| \\ &= (\|N_1\| + 2r\|N_2\|)\|I_1 - I_2\|. \end{aligned} \tag{B.10}$$

Thus the inequality

$$\|N_1\| + 2r\|N_2\| < 1 \tag{B.11}$$

holds and thus the contraction of F. It is not so difficult to check that (B.11) is satisfied if (B.8) holds. Hence we conclude [22] that the iteration procedure (43), (44) converges uniformly on $[0, d]$.

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